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Spectral protection and the geometry of recognition polytopes

Protección espectral y la geometría de los politopos de reconocimiento.

Artículo de investigación científica

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Resumen

El presente trabajo desarrolla la geometría espectral asociada al principio de reconocimiento establecido en el primer artículo de esta serie. A partir de la función de reconocimiento normalizada $R(H,K) = m(H,K)/N$, donde $m(H,K)$ representa el tamaño máximo de un emparejamiento compatible entre historias lineales uniformes, se introduce el Politopo de Reconocimiento P_k como el conjunto convexo de matrices simétricas de tamaño $k \times k$, con diagonal unitaria, cuyas entradas satisfacen las restricciones de subaditividad inducidas por la geometría de reconocimiento. Se demuestra que toda matriz de reconocimiento realizable pertenece a P_k, por lo que dicho politopo constituye una aproximación exterior del conjunto realizable R_k. Se estudia además la relación entre P_k y el cono C_k de matrices semidefinidas positivas. Se demuestra que P_3 está contenido en C_3 y se identifica un fenómeno de Protección Espectral en una familia cíclica simétrica de cuatro elementos, donde la frontera combinatoria coincide con la frontera espectral. Finalmente, se construye explícitamente un Hexágono Anómalo, una matriz perteneciente a P_6 pero no a C_6, mediante la exhibición de un autovalor negativo. Este resultado demuestra que las restricciones locales de reconocimiento no son suficientes para caracterizar la estructura global realizable. Se formula así el Principio de Obstrucción del Reconocimiento y se establecen problemas abiertos relacionados con la caracterización de R_k, la positividad semidefinida del núcleo de reconocimiento y la búsqueda de restricciones globales adicionales.
Palabras clave: geometría del reconocimiento; emparejamiento compatible; politopo de reconocimiento; matrices semidefinidas positivas; protección espectral; geometría métrica; hexágono anómalo.

Abstract

This work develops the spectral geometry associated with the recognition principle established in the first paper of this series. From the normalized recognition function $R(H,K) = m(H,K)/N$, where $m(H,K)$ denotes the maximum size of a compatible matching between uniform linear histories, we introduce the Recognition Polytope P_k as the convex set of symmetric $k \times k$ matrices with unit diagonal whose entries satisfy the subadditivity constraints induced by Recognition Geometry. We prove that every realizable recognition matrix belongs to P_k, making this polytope an outer approximation of the realizable set R_k. We then study the relationship between P_k and the cone C_k of positive semidefinite

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matrices. We prove that P_3 is contained in C_3 and identify a phenomenon of Spectral Protection in a symmetric cyclic four-point family, where the combinatorial boundary coincides with the spectral boundary. Finally, we construct explicitly an Anomalous Hexagon, a matrix belonging to P_6 but not to C_6, by exhibiting a negative eigenvalue. This result demonstrates that local recognition constraints are insufficient to characterize the global realizable structure. We therefore formulate the Recognition Obstruction Principle and identify open problems concerning the characterization of R_k, positive semidefiniteness of the recognition kernel, and the discovery of additional global constraints.
Keywords: recognition geometry; compatible matching; recognition polytope; positive semidefinite matrices; spectral protection; metric geometry; anomalous hexagon.

Introduction

In the first paper of this series (Sancho Lagla, 2026), it was established that the maximum compatible matching between uniform linear histories induces a natural metric structure on the space of histories. Let N be the structural size of a history and let m(H,K) denote the maximum size of a compatible matching between histories H and K. The normalized recognition function is defined by

$$\boldsymbol{R(H,K) = m(H,K)/N.}$$

The corresponding recognition distance is

$$\boldsymbol{D(H,K) = 1 - R(H,K).}$$

The fundamental result of the first paper establishes that the function D satisfies the triangle inequality. Expressed in terms of the recognition function, this property takes the form

$$\boldsymbol{R(H_i,H_l) \geq R(H_i,H_j) + R(H_j,H_l) - 1.}$$

This inequality constitutes the fundamental local law used throughout the present work.

A natural question then arises: what algebraic and spectral properties do the recognition matrices generated by this law possess? In particular, if one considers several histories $H_1, \ldots, H_k$ and constructs the matrix

$$\boldsymbol{G_{ij} = R(H_i,H_j),}$$

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is this matrix necessarily positive semidefinite?

The question is relevant because a positive semidefinite matrix can be interpreted as a matrix of inner products and therefore admits a geometric representation in a Hilbert space, following the framework of Aronszajn (1950) on reproducing kernels. If all recognition matrices were positive semidefinite, it would be possible to interpret the recognition function as a positive kernel and to develop a functional representation of the geometry.

However, the triangle inequality is a local condition. It is not evident that a set of constraints applied to triples of histories is sufficient to guarantee a global spectral property of a matrix of arbitrary dimension. Indeed, Vert (2008) has shown that kernels based on optimal assignment are not in general positive definite.

The objective of this paper is to study precisely this transition between local constraints and global spectral structure.

To this end, two fundamental objects are introduced. The first is the Recognition Polytope P_k, defined solely by the local recognition constraints. The second is the set R_k of matrices actually generated by families of histories. The former is accessible through linear inequalities; the latter depends on the combinatorial structure of compatible matchings.

The central problem can be expressed as

$$\boldsymbol{R_k \subseteq P_k.}$$

The fundamental question is to determine how much larger P_k can be than R_k, and what additional properties characterize the actually realizable matrices.

This paper obtains three main results. First, it is shown that the local constraints force positive semidefiniteness when $k = 3$. Second, a phenomenon of Spectral Protection is identified in a symmetric cyclic family of four elements. Third, an explicit matrix of six elements is constructed which satisfies all local constraints yet possesses a negative eigenvalue. This last construction constitutes the Anomalous Hexagon.

The result essentially modifies the interpretation of the polytope. P_k does not by itself represent the entire geometry of recognition. It constitutes an outer approximation, determined by local constraints, whereas the realizable geometry R_k may contain additional global constraints, in a line of research that connects with the metric geometry of Blumenthal (1953) and the cut polytopes studied by Deza and Laurent (1997).

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Methodology

The study employed a quantitative methodology with a descriptive approach, aimed at analyzing and characterizing the behavior of the recognition geometry and its associated spectral properties. To evaluate the proposed theoretical framework, three functional tests were conducted, allowing the verification of the main mathematical properties, constraints, and spectral behavior of the constructed recognition matrices.

The methodology employed is mathematical in nature and combines metric geometry, matrix theory, and spectral analysis. First, the starting point is the normalized recognition function R(H,K) and the subadditivity inequality obtained in the first paper of this series (Sancho Lagla, 2026).

Second, for a family of k histories, their recognition structure is represented by a symmetric matrix $G = (g_{ij})$, where

$g_{ij} = R(H_i,H_j)$.

The diagonal necessarily satisfies

$$g_{ii} = 1,$$

while the off-diagonal entries satisfy

$$0 \leq g_{ij} \leq 1.$$

The triangle inequality of the metric, expressed in terms of recognition, provides the constraints

$$g_{il} \geq g_{ij} + g_{jl} - 1.$$

From these conditions, the Recognition Polytope P_k is defined.

Third, the realizable set R_k is defined as the set of matrices that can actually be obtained from families of uniform histories.

Finally, positive semidefiniteness is analyzed through principal minors and eigenvalues. For symmetric cyclic families, advantage is also taken of the structure of circulant matrices, whose eigenvalues can be computed via the discrete Fourier transform, following the classical theory of Davis (1994).

This methodology makes it possible to distinguish clearly among three levels:

1. The local metric constraints.
2. The convex region P_k generated by those constraints.
3. The realizable subset R_k actually generated by histories.

The difference between these levels constitutes the central object of investigation. For the interpretation of the spectral results, recourse is made to the spectral graph theory of

Chung (1997) and to the foundational works of Schoenberg (1938) on metrics and positive definite functions.

Results

The Recognition Polytope

Definition 1. Recognition Polytope. For an integer $k \geq 2$, the Recognition Polytope P_k is the set of all symmetric matrices $G = (g_{ij}) \in \mathbb{R}^{k\times k}$ satisfying:

1. $g_{ii} = 1$ for all i.
2. $0 \leq g_{ij} \leq 1$ for all $i \neq j$.
3. $g_{il} \geq g_{ij} + g_{jl} - 1$ for all distinct indices i, j, l.

Because all these conditions are linear, P_k is a convex polytope.

The polytope is determined solely by the local recognition constraints. No additional condition derived from the internal structure of the histories is yet incorporated.

Realizable Subset

Definition 2. Realizable Subset. Let $H_1, \ldots, H_k$ be a collection of uniform linear histories. Their recognition matrix is defined by $G_{ij} = R(H_i,H_j)$. The set R_k of realizable matrices consists of all matrices that can be obtained in this way.

The relationship between both objects constitutes the first structural result.

Theorem 1. Realizable Inclusion. For all $k \geq 2$, $R_k \subseteq P_k$.

Proof. Every matrix belonging to R_k has entries of the form $G_{ij} = R(H_i,H_j)$. By the fundamental subadditivity result established by Sancho Lagla (2026), these entries satisfy $G_{il} \geq G_{ij} + G_{jl} - 1$. Moreover, by definition of the recognition function, $G_{ii} = R(H_i,H_i) = 1$ and $0 \leq G_{ij} \leq 1$. Consequently, every realizable matrix satisfies the conditions that define P_k. Hence, $R_k \subseteq P_k$. □

This result establishes that P_k is an outer approximation of the realizable set. The difference between the two sets is fundamental. The polytope contains all configurations compatible with the known local constraints, whereas R_k contains only those configurations that can actually arise from histories and compatible matchings.

Three Points: Recognition Constraints Imply PSD

Consider now the case $k = 3$. A matrix belonging to P_3 can be written as

$G = [1\ \ a\ \ b;\ a\ \ 1\ \ c;\ b\ \ c\ \ 1]$

where $0 \leq a, b, c \leq 1$ and the subadditivity constraints are

$a \geq b + c - 1$,

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$b \geq a + c - 1$,

$c \geq a + b - 1$.

Theorem 2. $P_3 \subseteq C_3$, where C_3 denotes the cone of positive semidefinite matrices of size 3×3.

Proof. To prove that G is positive semidefinite, its principal minors are considered. The first-order minors are 1, 1, 1, and are positive. The second-order principal minors are $1 - a^2$, $1 - b^2$, $1 - c^2$. Since $0 \leq a, b, c \leq 1$, all of them are non-negative. The determinant of G is $\det(G) = 1 + 2abc - a^2 - b^2 - c^2$. The subadditivity constraints prevent this determinant from being negative within P_3. In particular, on a subadditivity face, for example $b = a + c - 1$, one obtains $\det(G) = 2(a + c)(1 - a)(1 - c) \geq 0$. The remaining subadditivity faces are equivalent by permutation of the variables. On the faces corresponding to the bounds of the constraint cube, the subadditivity condition likewise reduces the domain so that the determinant remains non-negative. Consequently, all principal minors of G are non-negative. By Sylvester's criterion for symmetric matrices, $G \geq 0$ in the semidefinite sense. Therefore, $P_3 \subseteq C_3$. □

This result demonstrates that, for three points, the local recognition constraints are sufficiently strong to guarantee a global spectral property.

Spectral Protection in a Cyclic Four-Point Family

The behavior for four points makes it possible to observe an additional phenomenon. Consider a symmetric cyclic family whose matrix has the form

$G_4 = [1\ \ a\ \ b\ \ a;\ a\ \ 1\ \ a\ \ b;\ b\ \ a\ \ 1\ \ a;\ a\ \ b\ \ a\ \ 1]$.

Its eigenvalues are:

$\lambda_0 = 1 + 2a + b$,

$\lambda_1 = 1 - b$,

$\lambda_2 = 1 - 2a + b$,

$\lambda_3 = 1 - b$.

The only eigenvalue that can become negative is $\lambda_2 = 1 - 2a + b$. The condition $\lambda_2 < 0$ is equivalent to $b < 2a - 1$. However, the subadditivity constraint applied to the corresponding triple yields precisely $b \geq 2a - 1$. Hence, $\lambda_2 \geq 0$. The boundary $b = 2a - 1$ is simultaneously a boundary of the Recognition Polytope and a boundary of the cone of positive semidefinite matrices.

This exact alignment constitutes the phenomenon that we term Spectral Protection.

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Definition 3. Spectral Protection. A family F contained in P_k exhibits Spectral Protection when the local recognition constraints guarantee that all matrices of F are positive semidefinite and, moreover, the boundary determined by those constraints coincides with the spectral boundary within the family considered.

The symmetric cyclic family of four points constitutes an explicit example of this phenomenon. The importance of this result lies in the fact that a local combinatorial inequality, derived from the matching problem, directly prevents the appearance of a negative spectral mode.

The Anomalous Hexagon

The protection observed in small dimensions is not universal. Consider now a symmetric cyclic family of six points with first row (1, a, b, c, b, a). We choose a = 5/6, b = 4/6, c = 5/6.

The corresponding matrix G_6 is the symmetric circulant matrix with that first row.

Lemma 1. Membership in P_6. $G_6 \in P_6$.

Proof. By cyclic symmetry and the action of the dihedral group D_6, the triangle inequalities can be reduced to seven non-trivial classes:

$b \geq 2a - 1$,

$c \geq a + b - 1$,

$b \geq a + c - 1$,

$a \geq b + c - 1$,

$c \geq 2a - 1$,

$c \geq 2b - 1$,

$a \geq 2c - 1$.

Substituting a = c = 5/6 and b = 4/6, all inequalities are satisfied. Some are attained exactly. Therefore, $G_6 \in P_6$.

Theorem 3. Spectral Obstruction of the Hexagon. $P_6 \not\subset C_6$. That is, there exist matrices belonging to the Recognition Polytope that are not positive semidefinite.

Proof. Since G_6 is a symmetric circulant matrix, its eigenvalues can be computed via the discrete Fourier transform (Davis, 1994). For a symmetric circulant matrix with first row (1, a, b, c, b, a), the eigenvalues are

$\lambda_k = 1 + 2a \cos(2\pi k/6) + 2b \cos(4\pi k/6) + c \cos(\pi k)$.

Evaluating for k = 0, 1, 2, 3, 4, 5 we obtain:

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$\lambda_0 = 1 + 2a + 2b + c = 29/6,$

$\lambda_1 = 1 + a - b - c = 1/3,$

$\lambda_2 = 1 - a - b + c = 1/3,$

$\lambda_3 = 1 - 2a + 2b - c = -1/6,$

$\lambda_4 = 1 - a - b + c = 1/3,$

$\lambda_5 = 1 + a - b - c = 1/3.$

Thus, the spectrum of G_6 is $\{29/6, 1/3, 1/3, -1/6, 1/3, 1/3\}$. There exists a strictly negative eigenvalue: $\lambda_3 = -1/6 < 0$. Consequently, $G_6 \notin C_6$. We thus obtain that $P_6 \not\subset C_6$.

The matrix G_6 constitutes the Anomalous Hexagon. Its importance does not lie solely in the existence of a negative eigenvalue. The result demonstrates that all local recognition constraints can be satisfied simultaneously while a global positivity property fails. Therefore, local constraints do not completely characterize the global geometry.

4. Discussion

The results obtained reveal a hierarchy of spectral constraints induced by the geometry of recognition. The starting point is the local recognition law established in the first paper of this series, where the normalized recognition function gives rise to a distance satisfying the triangle inequality (Sancho Lagla, 2026). The present results show, however, that the consequences of that local law depend strongly on the number and organization of the histories under consideration.

For $k = 3$, the local constraints are sufficiently strong to guarantee positive semidefiniteness, so that $P_3 \subseteq C_3$. This is a particularly significant result because positive semidefiniteness is precisely the algebraic condition that permits a matrix to be interpreted as a Gram matrix and, consequently, as a system of inner products in a Hilbert-space representation. In this sense, the three-history case is compatible with the classical framework of positive kernels developed by Aronszajn (1950). It is also consistent with the broader relation between metric structure and positive definiteness studied by Schoenberg (1938), who showed that metric and kernel properties are linked through nontrivial global conditions rather than through arbitrary pairwise similarities alone.

The symmetric cyclic family of four points provides a second level of structure. In that family, the phenomenon identified here as Spectral Protection shows that the boundary imposed by the recognition inequalities prevents the parameters from entering the region where a negative eigenvalue would occur. Because the matrix possesses a cyclic symmetry, its spectral behavior can be understood in the context of structured matrix

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theory, particularly the analysis of circulant matrices described by Davis (1994). More broadly, the fact that global information can be extracted from symmetry and eigenvalue structure is closely related to the perspective of spectral graph theory, in which combinatorial organization is encoded through spectra (Chung, 1997). The important point in the present setting is that symmetry produces an additional form of protection that is not visible from the triangle inequalities considered separately.

However, the Anomalous Hexagon demonstrates that this protection is not universal. The fundamental relationship among the three objects studied can be summarized as $R_k \subseteq P_k$, while the Anomalous Hexagon establishes that $P_6 \not\subset C_6$. This does not imply, by itself, that $R_6 \not\subset C_6$. The anomalous matrix was constructed inside P_6, not inside R_6. Therefore, the correct interpretation is that the local inequalities defining P_k are insufficient to guarantee positive semidefiniteness, but the construction does not yet prove that an actually realizable recognition matrix can be indefinite.

This distinction is essential. The Recognition Polytope P_k should be understood as a convex outer approximation of the configurations that satisfy the local recognition constraints, whereas R_k represents the subset that can actually arise from families of histories and their compatible matchings. From the viewpoint of distance geometry, this separation between formally admissible distance data and geometrically realizable configurations is natural: Blumenthal (1953) emphasized that global realizability generally requires conditions that cannot be reduced to isolated local distance relations. A similar phenomenon appears in the geometry of cuts and metrics, where systems of elementary metric inequalities define polyhedral relaxations, but additional facets or global combinatorial restrictions are needed to characterize more structured metric objects (Deza & Laurent, 1997). Thus, the gap between P_k and R_k is not merely a technical inconvenience; it is the central geometric problem generated by the recognition formalism.

The Anomalous Hexagon also has an important interpretation from kernel theory. If the recognition function were automatically a positive definite kernel for arbitrary families of histories, every recognition matrix would be positive semidefinite. The six-element counterexample at the level of P_6 shows that such positivity cannot be derived from the local recognition inequalities alone. This conclusion is consistent with the warning provided by Vert (2008), who demonstrated that optimal-assignment-based similarities are not positive definite in general. Although the recognition construction studied here is

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not identical to the optimal assignment kernel considered by Vert, both settings illustrate the same structural difficulty: optimizing pairwise correspondences does not automatically guarantee that all pairwise optima can be embedded simultaneously into a single positive semidefinite global geometry.

This observation leads to the following principle. Recognition Obstruction Principle. Local recognition compatibility is not sufficient to characterize the global geometry of recognition. The Anomalous Hexagon therefore constitutes an explicit obstruction to any theory that identifies the Recognition Polytope directly with the realizable recognition set. Its conceptual consequence is stronger than the mere existence of one negative eigenvalue. The example shows that pairwise and triplewise admissibility can coexist with a global spectral inconsistency. Hence, any complete characterization of R_k must contain constraints that control the simultaneous compatibility of several maximum matchings, rather than only the numerical values of recognition attached to individual pairs or triples.

One possible interpretation is that realizability requires a form of higher-order coherence. Each entry $R(H_i,H_j)$ records an optimal pairwise relation, but a matrix of such optima does not record whether the corresponding maximizing matchings can coexist within a common global combinatorial structure. This distinction between pairwise optimality and simultaneous consistency is precisely where hidden inequalities may arise. In polyhedral terms, such inequalities would define additional faces separating R_k from the larger polytope P_k; in spectral terms, they would prevent locally admissible configurations from crossing into regions where the smallest eigenvalue becomes negative.

The results also suggest that the threshold at which local constraints cease to control the spectrum is itself a meaningful invariant of the theory. The cases $k = 3$, $k = 4$ under cyclic symmetry, and $k = 6$ exhibit three qualitatively different regimes: unconditional local protection, symmetry-assisted protection, and failure of local protection. Determining what happens for $k = 4$ without symmetry and for $k = 5$ becomes therefore an important intermediate problem. Such cases may reveal whether the Anomalous Hexagon is the first possible obstruction in cardinality or merely the first explicit obstruction currently identified.

A second open direction concerns the structure of the negative eigenspace. The anomalous matrix proves indefiniteness, but the eigenvector associated with the negative eigenvalue may contain additional combinatorial information. Its sign pattern could

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indicate which subsets of histories compete globally and which collections of local recognition relations cannot be realized simultaneously. This type of spectral interpretation would connect naturally with methods in spectral graph theory (Chung, 1997), where eigenvectors frequently encode nonlocal combinatorial structure, and with structured matrix analysis (Davis, 1994), where symmetry can make such spectral mechanisms explicit.

A third direction is to search for realizability inequalities that are invisible at the level of triples. The analogy with cut and metric polytopes suggests that higher-order inequalities may be necessary (Deza & Laurent, 1997). In the recognition setting, these inequalities should not be introduced abstractly; ideally, they should arise from the combinatorics of compatible matchings themselves. Such a derivation would explain why the local triangle-type law is necessary but not sufficient and would provide a genuine intrinsic description of R_k rather than an external semidefinite approximation.

Finally, the distinction between local recognition constraints and global realizability provides a natural transition toward the next stage of the research programme. The theory of geodesic factorization and recognition transport developed subsequently by Sancho Lagla (2025) studies how recognition relations propagate and compose through intermediate histories. From the perspective established here, transport and composition are not merely additional operations on histories; they are potential mechanisms for expressing the higher-order coherence that is absent from the definition of P_k. If transport laws impose compatibility among several pairwise maximizing relations at once, they may supply precisely the type of global constraints required to distinguish R_k from P_k.

Taken together, these results indicate that the geometry of recognition is neither purely metric nor purely spectral. Its local metric law generates a tractable polytope, but the realizable theory appears to require an additional combinatorial layer governing the simultaneous coexistence of matchings. The three-point theorem shows that local information can be sufficient in low dimension; the cyclic four-point family shows that symmetry can reinforce this sufficiency; and the Anomalous Hexagon shows that, in higher dimension, local admissibility can diverge from spectral admissibility. The central problem for the continuation of the programme is therefore to identify the missing global laws and determine whether those laws ultimately restore positive semidefiniteness for all realizable recognition matrices or instead lead to genuine indefinite configurations inside R_k. Resolving this alternative would determine whether recognition geometry

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admits a universal Hilbert-space realization in the sense suggested by Aronszajn (1950) and Schoenberg (1938), or whether its natural global geometry belongs to a broader, intrinsically non-Hilbertian class.

Conclusions

This paper has established a first description of the spectral geometry associated with the recognition principle. The Recognition Polytope P_k was introduced as the set of symmetric matrices with unit diagonal that satisfy the subadditivity constraints derived from the recognition metric. It was proved that every realizable matrix belongs to this polytope: $R_k \subseteq P_k$. It was also proved that, for three points, $P_3 \subseteq C_3$, showing that local recognition constraints can impose global positive semidefiniteness in small dimensions. For a symmetric cyclic family of four points, the phenomenon of Spectral Protection was identified, where the combinatorial boundary coincides with the spectral boundary. Finally, the Anomalous Hexagon was constructed, an explicit matrix that satisfies the constraints of P_6 but possesses the eigenvalue $\lambda_3 = -1/6$. Consequently, $P_6 \not\subset C_6$. This result constitutes the first explicit spectral obstruction of the theory. It demonstrates that the geometry defined exclusively by the local recognition constraints cannot be the complete global geometry.

The fundamental problem is thus reformulated: it is not merely a matter of studying P_k, but of characterizing the realizable subset R_k within P_k. Among the principal open questions are the characterization of the vertices of P_k, in order to determine the extreme points of the Recognition Polytope and whether they possess a direct combinatorial interpretation in terms of histories; the determination of the transition dimension, that is, the smallest value of k for which P_k is not contained in C_k, noting that the present work demonstrates that k = 6 is sufficient while the complete characterization for intermediate dimensions remains open; the identification of additional recognition constraints beyond subadditivity imposed by the combinatorial structure of compatible matchings; the characterization of R_k through a complete mathematical description of the realizable subset of P_k; and the positivity of the recognition kernel, in order to determine whether $R_k \subseteq C_k$ for all k. An affirmative answer would imply that the recognition kernel possesses a representation via inner products in a Hilbert space, in the spirit of the works of Schoenberg (1938) and Aronszajn (1950). A negative answer would mean that the recognition kernel is indefinite and that the global geometry would require a more general structure. In both cases, the Anomalous Hexagon demonstrates that the solution cannot

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be obtained solely from the local constraints. The study of global recognition propagation therefore constitutes the next natural step of this mathematical programme.

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Declaración

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