SERCAPO JOURNAL SCIENTIFIC
Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206
DOI: https://doi.org/10.67166/c9cv5070

Recognition geometry I: maximum compatible matchings induce a pseudometric on uniform linear histories

Geometría del reconocimiento I: los emparejamientos compatibles máximos inducen una pseudométrica sobre historias lineales uniformes.

Artículo de investigación científica
Ciencias de la Educación

Jorge Esteban Sancho Lagla
jorge19esteban@gmail.com
https://orcid.org/0009-0006-4604-8984
Universidad Técnica de Ambato
Facultad de Ciencias de la Educación

Fecha de envío: 2026-06-07 **Fecha de revisión:** 2026-07-02 **Fecha da aceptación:** 2026-08-01

Resumen.

El objetivo de esta investigación fue demostrar que el complemento de la similitud normalizada, obtenida mediante emparejamientos compatibles máximos, define una pseudométrica sobre el espacio de las historias lineales uniformes. Estas historias se concibieron como grafos dirigidos acíclicos finitos, conformados por uniones disjuntas de caminos dirigidos, con vértices tipificados y una cardinalidad común. Metodológicamente, se desarrolló una investigación básica con enfoque cuantitativo, teórico y deductivo, bajo un diseño no experimental. El procedimiento comprendió la definición formal de emparejamientos estrictamente realizables, sujetos a conservación de tipos, inyectividad e isomorfismo de orden entre los subposets emparejados. Posteriormente, se maximizó su cardinalidad para construir una función de similitud y se definió la distancia de reconocimiento como su complemento. Los resultados demostraron que esta distancia satisface no negatividad, simetría, distancia propia nula y desigualdad triangular. Esta última propiedad se estableció mediante un lema fundamental de subaditividad, basado en la composición de emparejamientos máximos y el principio de inclusión-exclusión. Asimismo, se construyó el espacio cociente métrico denominado Espacio de Reconocimiento y se comprobó que sus clases de equivalencia corresponden exactamente a las clases de isomorfismo de los grafos dirigidos tipificados. Un ejemplo con tres historias de cardinalidad tres confirmó el comportamiento esperado de las similitudes y distancias calculadas. Se concluye que la optimización combinatoria propuesta genera una estructura geométrica rigurosa para comparar historias lineales uniformes. Este marco proporciona bases para futuras investigaciones sobre algoritmos computacionales, representaciones espectrales y la semidefinición positiva de la función de similitud, y fortalece el análisis matemático de estructuras causales discretas.

Palabras clave: historias lineales uniformes; emparejamiento compatible máximo; pseudométrica; grafos dirigidos tipificados; función de similitud.

Abstract.

The objective of this study was to demonstrate that the complement of the normalized similarity obtained through maximum compatible matchings defines a pseudometric on the space of uniform linear histories. These histories were conceived as finite directed acyclic graphs composed of disjoint unions of directed paths, with typed vertices and a common cardinality. Methodologically, basic research was conducted using a quantitative, theoretical, and deductive approach under a nonexperimental design. The procedure involved formally defining strictly realizable matchings subject to type preservation, injectivity, and order isomorphism between matched subposets. Their cardinality was then maximized to construct a similarity function, and recognition distance was defined as its complement. The results demonstrated that this distance satisfies nonnegativity, symmetry, zero self-distance, and the triangle inequality. The latter property was established through a fundamental subadditivity lemma based on the composition of maximum matchings and the inclusion-exclusion principle. Furthermore, the metric quotient called the Recognition Space was constructed, and its equivalence classes were shown to correspond exactly to the isomorphism classes of typed directed graphs. An illustrative example involving three histories of cardinality three confirmed the expected behavior of the calculated similarities and distances. It is concluded that the proposed combinatorial optimization problem generates a rigorous geometric structure for comparing uniform linear histories. This framework provides a mathematical foundation for future research on computational algorithms, spectral representations, functional embeddings, and the positive semidefiniteness of the recognition similarity function, thereby extending optimization-induced geometry to a new class of discrete combinatorial objects within contemporary mathematical and computational research.

Keywords: uniform linear histories; maximum compatible matching; pseudometric; typed directed graphs; similarity function.

Introduction

A recurring and influential theme in modern mathematics is the principle that optimization problems defined over pairs of objects can induce meaningful geometric and metric structures. These constructions make it possible to quantify differences between complex objects while preserving their essential structural properties. Optimal transport provides a representative example: the Wasserstein distance emerges from minimizing the transportation cost between probability distributions and has become an important mathematical tool for comparing structured data (Peyré & Cuturi, 2019). Its geometric interpretation has also been extended to comparisons between metric-measure spaces through the Gromov Wasserstein framework (Mémoli, 2011).

Related developments demonstrate that optimization-induced distances can be applied beyond classical metric spaces. For instance, the Gromov–Wasserstein distance between networks enables the comparison of weighted and directed structures while remaining stable under appropriate perturbations (Chowdhury & Mémoli, 2020). These

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730

contributions illustrate how an optimization problem can provide the metric foundation upon which topological, analytical, and computational theories are subsequently constructed.

A similar principle appears in structural pattern recognition, where graph edit distance quantifies dissimilarity through an optimal sequence of vertex and edge insertions, deletions, and substitutions. The close relationship between edit distance and maximum common subgraph problems demonstrates that structural similarity can be expressed through either minimum transformation cost or maximum shared structure (Bunke & Shearer, 1998). Graph edit distance has therefore become a central approach for error-tolerant graph matching and the comparison of relational structures (Gao et al., 2010).

Graph matching research has produced a broad collection of exact, approximate, and heuristic procedures for identifying structural correspondences between graphs (Conte et al., 2004). In particular, subgraph-isomorphism algorithms establish mappings that preserve adjacency relations and allow smaller structural patterns to be recognized within larger graphs (Cordella et al., 2004). These approaches confirm that the selection of an appropriate correspondence condition is fundamental: a weak condition may admit structurally inconsistent alignments, whereas an excessively restrictive one may prevent meaningful partial comparisons.

Maximum common subgraph constructions are especially relevant to metric geometry because they directly measure the largest structure shared by two objects. Bunke and Shearer (1998) demonstrated that an appropriately normalized maximum common subgraph can generate a graph distance metric. This result motivates the broader question of whether other maximum-compatible-substructure problems can also induce valid metric or pseudometric spaces.

The present paper introduces a new member of this family of optimization-induced geometries. We investigate a class of combinatorial objects termed uniform linear histories: finite, typed, directed acyclic graphs composed of disjoint unions of directed paths, with all graphs under consideration sharing the same number of vertices. Directed paths encode internal causal or precedence relations, whereas vertex types represent the local categories that must be respected during structural comparison.

A natural question then arises: what is the maximum proportion of one history that can be consistently aligned with another without violating their internal topological orders? We formulate this question as a maximum compatible matching problem. A matching is

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206

considered strictly realizable when it preserves vertex types, remains injective, and acts as a partial order isomorphism between the matched subposets. Therefore, the matching must preserve and reflect the order relation. This biconditional requirement also preserves incomparability, preventing vertices located on independent paths from being incorrectly aligned with causally related vertices.

By maximizing the cardinality of a strictly realizable matching, we obtain a normalized measure of structural similarity. This construction is related to the wider literature on graph similarities and graph kernels, in which functions defined over pairs of graphs are used to capture common substructures and relational patterns (Kriege et al., 2020; Nikolentzos et al., 2021). However, the present framework is specifically based on strict order-compatible matchings and does not initially assume the existence of a vector-space representation or a positive semidefinite kernel.

The primary contribution of this paper is to prove that the complement of the maximized similarity defines a natural pseudometric on the space of uniform linear histories. The main mechanism underlying this result is a fundamental subadditivity lemma governing the cardinality of maximum compatible matchings under composition. This lemma leads directly to the triangle inequality, while symmetry, non-negativity, and zero self-distance follow from the basic properties of the matching problem.

Because a pseudometric may assign zero distance to distinct but structurally equivalent objects, we subsequently construct a metric quotient called the Recognition Space. We prove that its points correspond exactly to the isomorphism classes of uniform linear histories considered as typed directed graphs. Consequently, two histories have zero recognition distance precisely when a type-preserving directed graph isomorphism exists between them.

This quotient construction is consistent with the broader mathematical goal of representing combinatorial objects through invariant geometric structures. Related stability results in topological data analysis show how suitable distances can preserve meaningful information while remaining insensitive to controlled perturbations (Cohen-Steiner et al., 2007). In the present setting, however, the geometry is generated directly by exact structural compatibility rather than by a continuous deformation or filtration.

The current study is devoted entirely to establishing this discrete metric geometry. It does not yet examine linear embeddings, spectral representations, or reproducing-kernel properties. Graph kernel theory shows that positive semidefiniteness is essential when a

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206

similarity function is intended to represent an inner product in a Hilbert space (Kriege et al., 2020; Nikolentzos et al., 2021). Accordingly, determining whether the proposed recognition similarity is positive semidefinite remains an open problem. The pseudometric and its associated quotient space provide the rigorous geometric foundation required for subsequent spectral, functional, and computational investigations.

Objective

To demonstrate that the complement of the similarity obtained through maximum compatible matchings defines a pseudometric on the space of uniform linear histories.

Methodology

The research was conducted using a quantitative, theoretical, and deductive approach aimed at the construction and formal demonstration of a geometric structure applicable to discrete spaces. This was a basic research study, as it sought to advance mathematical knowledge concerning the relationship between combinatorial optimization problems and metric structures. It also followed a non-experimental design, since no empirical variables were manipulated and no participant population was involved.

The units of analysis consisted of the so-called uniform linear histories, defined as finite directed acyclic graphs formed by disjoint unions of directed paths. Their vertices were assigned specific types, and all the histories under consideration shared a fixed cardinality N. The set comprising these structures was denoted by HN.

Matchings

Let P be a countably infinite set of ports (vertices) and T a finite set of types. Each port p $\in$ P is assigned a type via a fixed mapping $\tau : P \to T$.

Definition 2.1 (Uniform Linear History) . A uniform linear history H

is a finite directed acyclic graph (VH, EH) such that:

(1) VH is a finite subset of P .

(2) The graph is a disjoint union of finitely many directed paths.

(3) All histories considered within a given context are uniform; that is, they share a fixed cardinality N = |VH |.

This integer N acts as the intrinsic structural scale of the space and serves as our normalization constant. Let *HN* denote the set of all uniform linear histories of size N .

To define a valid structural alignment between two histories, the matching must respect both the local identities (types) and the global topology (causal order). Let $\preceq$H denote the

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206

partial order induced by reachability in the DAG H (i.e., u $\preceq_H$ v if and only if there exists a directed path from u to v).

Definition 2.2 (Strictly Realizable Matching). A matching between two histories H, K ∈ HN is a set of pairs $\mu \subseteq VH \times VK$ such that:

(1) Type Preservation: $\tau(p) = \tau(q)$ for all $(p, q) \in \mu$.

(2) Injectivity: The natural projections of μ onto VH and VK are injective.

(3) Order-Isomorphism: For any two pairs $(p1, q1), (p2, q2) \in \mu$, the following biconditional holds:

$$p1 \preceq_H p2 \iff q1 \preceq_K q2.$$

This biconditional constraint is the mathematical heart of the theory. It demands that the matching both preserves and reflects the partial order. Crucially, it ensures that incomparability is also strictly preserved: ports residing on parallel, independent paths in H must map to ports on parallel, independent paths in K.

Let M(H, K) denote the set of all strictly realizable matchings between H and K. The maximum compatible matching size is defined

as:

$$m(H, K) = \max_{\mu \in M(H,K)} |\mu|.$$

Since the identity map trivially satisfies the biconditional, $m(H, H) = |VH| = N$.

The recognition similarity function

Definition 3.1 (Recognition Similarity). For two uniform linear histories H, K ∈ HN , their recognition similarity is given by:

$$R(H, K) = \frac{m(H, K)}{N}.$$

Proposition 3.2 (Basic Properties). For any uniform linear histories H, K ∈ HN :

(1) $R(H, K) = R(K, H)$ *(Symmetry)* .

(2) $R(H, H) = 1$ *(Unit diagonal).*

(3) $0 \leq R(H, K) \leq 1$ *(Boundedness).*

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730

Proof. Symmetry follows from the symmetry of the conditions in Definition 2.2. The unit diagonal follows from $m(H, H) = N$. Boundedness holds because the maximum matching size cannot exceed the total number of vertices N, yielding $0 \leq m \leq N$.

Fundamental subadditivity and the recognition pseudometric

To establish a geometric structure, we define a distance function based on the structural discrepancy.

Definition 4.1 (Recognition Distance). The function $D : HN \times HN \to$ R is defined as:

$$D(H, K) = 1 - R(H, K) = \frac{N - m(H, K)}{N}.$$

The viability of D as a pseudometric relies entirely on the metric behavior of the optimization problem under composition. We establish this via the following fundamental subadditivity lemma, which serves as the core engine of the geometry.

Lemma 4.2 (Fundamental Subadditivity). For any three uniform linear histories H1, H2, H3 ∈ HN

$$m(H1, H3) \geq m(H1, H2) + m(H2, H3) - N.$$

Proof. Let $\mu_{12} \in M(H1, H2)$ and $\mu_{23} \in M(H2, H3)$ be maximum realizable matchings of sizes m_{12} and m_{23}, respectively. Define the set of "bridge" ports in H2 as $B = Im(\mu_{12}) \cap Dom(\mu_{23})$. By the inclusion-exclusion principle and the cardinality bound N,

$|B| \geq |Im(\mu 12)| + |Dom(\mu_{23})| - |VH2| = m_{12} + m_{23} - N.$

For each $q \in B$, there exist unique ports $p \in VH1$ and $r \in VH3$ such that $(p, q) \in \mu_{12}$ and $(q, r) \in \mu_{23}$. We construct the composite set $\mu_{13} = \{(p, r) : q \in B\}$. We must verify that $\mu_{13} \in M(H1, H3)$.

First, type preservation holds: since $\tau(p) = \tau(q)$ and $\tau(q) = \tau(r)$, it follows that $\tau(p) = \tau(r)$. Second, injectivity is inherited directly from the injectivity of μ12 and μ23. Finally, for the order-isomorphism, consider any two pairs (p1, r1), (p2, r2) ∈ μ13 generated by bridge ports q1, q2 ∈ B. By the transitivity of the biconditional:

$$p1 \preceq_{H1} p2 \iff q1 \preceq_{H2} q2 \iff r1 \preceq_{H3} r2.$$

The first equivalence holds because μ12 is realizable, and the second because μ23 is realizable. Thus, μ13 satisfies Definition 2.2. Since μ13 is a strictly realizable matching, the maximum matching size m(H1, H3) is at least |μ13| = |B|. Therefore, $m(H1, H3) \geq m_{12} + m_{23} - N$.

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730

Theorem 4.3 (Recognition Pseudometric Theorem) . The function

D(H, K) is o pseudometric on the space HN .

Proof. We verify the metric axioms.

(1) Non-negativity: D(H, K) ≥ 0 since R(H, K) ≤ 1.

(2) Self-distance: D(H, H) = 1 − 1 = 0.

(3) Symmetry: D(H, K) = D(K, H) follows from Proposition 3.2.

(4) Triangle Inequality: We must show D(H1, H3) ≤ D(H1, H2) + D(H2, H3). Using the Definition 4.1, the triangle inequality is equivalent to the inequality established in Lemma 4.2.

The recognition space (quotient metric space)

The pseudometric D(H, K) = 0 implies m(H, K) = N . We now formally link this extreme optimization value to the structural identity of the graphs.

Lemma 5.1 (Order Isomorphism to Graph Isomorphism) . Let H, K ∈ HN . If there exists a strictly realizable matching of size N between H and K, then H and K are isomorphic as typed directed graphs.

Proof. A matching μ of size N implies a bijection *μ : VH → VK*

that acts as a full order-isomorphism on the posets *(VH, ⪯H) and (VK, ⪯K)*. Because H and K are disjoint unions of directed paths, their directed edges are exactly the covering relations of their respective posets (i.e., u → v ∈ EH if u ≺H v and there is no intermediate w). Since an order-isomorphism strictly preserves covering relations, (u, v) ∈ EH ⇐⇒ (μ(u), μ(v)) ∈ EK. As μ also preserves types by definition, it constitutes a valid graph isomorphism.

Corollary 5.2 (The Recognition Space). The relation *H ∼ K ⇐⇒ D(H, K) = 0* is an equivalence relation. The quotient space HN /∼, equipped with the metric D˜([H], [K]) = D(H, K), is a well-defined metric space. Furthermore, the points of this space correspond exactly to the isomorphism classes of uniform linear histories.

We define this metric quotient *(HN /∼, D˜)* as the Recognition Space. It provides a rigorous topological space where proximity reflects the exact size of the maximum compatible sub-structure shared by the graphs.

Illustrative example

Consider three uniform linear histories of size N = 3, consisting of single directed paths. The letters denote distinct port types:

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206

• $H1 = (A \to B \to C)$

• $H2 = (A \to D \to C)$

• $H3 = (B \to D \to E)$

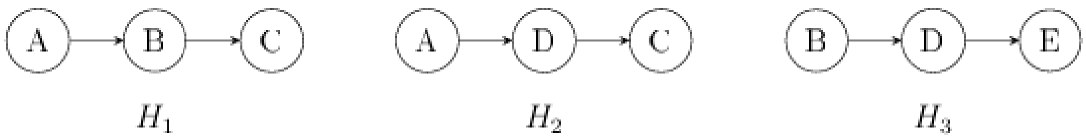

Figure 1. Three uniform linear histories of size 3.

Because the graphs are simple paths, the order condition reduces strictly to preserving the sequence of matched types.

• H1 and H2: Share types A and C. The matching {(A, A), (C, C)} respects the internal order ($A \prec C$ in both). Thus, m12 = 2.

• H2 and H3: Share only type D. Thus, m23 = 1.

• H1 and H3: Share only type B. Thus, m13 = 1 (matching B with B is trivially order-preserving, as a matching of size 1 has no pairs to violate the biconditional).

The similarities are R12 = 2/3, R_{23} = 1/3, and R13 = 1/3. The corresponding distances are D12 = 1/3, D_{23} = 2/3, and D13 = 2/3. The triangle inequality is strictly satisfied (e.g., $D13 \leq D12 + D23 \implies 2/3 \leq 1/3 + 2/3$).

Discussion

In this paper, we have demonstrated that the discrete combinatorial problem of finding a maximum compatible matching under strict bi-conditional order constraints is not merely an algorithmic task, but a geometric generator. It induces a natural pseudometric that classifies linear histories up to graph isomorphism, establishing the Recognition Space.

This work sits squarely within the paradigm where optimization induces geometry, providing a rigid structural counterpart to metrics like the edit distance or Gromov-Hausdorff distance. However, by establishing this metric quotient, we arrive at a deeper, unresolved mathematical question regarding the linear representability of this space.

It is a well-known phenomenon in graph theory that similarity measures based on maximum assignments often fail to be positive semi-definite (PSD) due to the non-linear nature of the maximum operator. However, the restrictive biconditional order-isomorphism required by Definition 2.2 severely limits the matching space, preserving

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206

poset structure in ways that standard assignment kernels do not. This leads to our central open problem.

Open Problem 7.1. Determine whether the Recognition Similarity function R(H, K) = m(H, K)/N defines a positive semidefinite kernel on the Recognition Space.

Whether R is ultimately proven to be positive semidefinite (permitting an isometric embedding into a Reproducing Kernel Hilbert Space) or indefinite (necessitating pseudo-Hilbert spaces such as Krein spaces), the Recognition Space (HN /∼, D˜) remains mathematically robust as a metric object.

This paper constitutes strictly the geometric foundation of the theory. The resolution of the Open Problem, alongside the study of the spectral geometry of recognition matrices and their analytical properties, forms a dedicated research program to be developed in subsequent papers.

Conclusions

The findings demonstrate that the complement of the normalized similarity obtained through maximum compatible matchings defines a valid pseudometric on the space of uniform linear histories. The proof is supported by the fundamental subadditivity lemma, which establishes the behavior of maximum matchings under composition and guarantees the triangle inequality. Together with non-negativity, symmetry, and zero self-distance, this result confirms that a discrete combinatorial optimization problem can generate a mathematically consistent geometric structure for comparing typed directed acyclic graphs while preserving their internal causal and topological organization.

The construction of the metric quotient, designated as the Recognition Space, provides a rigorous framework for organizing uniform linear histories according to their structural equivalence. The study demonstrates that its equivalence classes correspond exactly to the isomorphism classes of typed directed graphs; therefore, two histories have zero recognition distance precisely when they possess the same type-preserving structural organization. This geometric framework establishes a solid basis for future studies on spectral properties, functional representations, computational algorithms, and the possible positive semidefiniteness of the recognition similarity function.

References

Bunke, H., & Shearer, K. (1998). A graph distance metric based on the maximal common subgraph. *Pattern Recognition Letters*, *19*(3–4), 255–259. https://doi.org/10.1016/S0167-8655(97)00179-7

Chowdhury, S., & Mémoli, F. (2020). The Gromov–Wasserstein distance between networks and stable network invariants. *Information and Inference: A Journal of the IMA, 9*(2), 419–475. https://doi.org/10.1093/imaiai/iaz026

Cohen-Steiner, D., Edelsbrunner, H., & Harer, J. (2007). Stability of persistence diagrams. *Discrete & Computational Geometry, 37*(1), 103–120. https://doi.org/10.1007/s00454-006-1276-5

Conte, D., Foggia, P., Sansone, C., & Vento, M. (2004). Thirty years of graph matching in pattern recognition. *International Journal of Pattern Recognition and Artificial Intelligence, 18*(3), 265–298. https://doi.org/10.1142/S0218001404003228

Cordella, L. P., Foggia, P., Sansone, C., & Vento, M. (2004). A (sub)graph isomorphism algorithm for matching large graphs. *IEEE Transactions on Pattern Analysis and Machine Intelligence, 26*(10), 1367–1372. https://doi.org/10.1109/TPAMI.2004.75

Gao, X., Xiao, B., Tao, D., & Li, X. (2010). A survey of graph edit distance. *Pattern Analysis and Applications, 13*(1), 113–129. https://doi.org/10.1007/s10044-008-0141-y

Kriege, N. M., Johansson, F. D., & Morris, C. (2020). A survey on graph kernels. *Applied Network Science, 5*, Article 6. https://doi.org/10.1007/s41109-019-0195-3

Mémoli, F. (2011). Gromov–Wasserstein distances and the metric approach to object matching. *Foundations of Computational Mathematics, 11*(4), 417–487. https://doi.org/10.1007/s10208-011-9093-5

Nikolentzos, G., Siglidis, G., & Vazirgiannis, M. (2021). Graph kernels: A survey. *Journal of Artificial Intelligence Research, 72*, 943–1027. https://doi.org/10.1613/jair.1.13225

Peyré, G., & Cuturi, M. (2019). Computational optimal transport. *Foundations and Trends in Machine Learning, 11*(5–6), 355–607. https://doi.org/10.1561/2200000073

Vol. 3 Núm. 3 (2026): Edición Frecuencia Trimestral de Julio/Septiembre, pp 718-730
ISNN: 3151-8206

Declaración

Declaración	Detalle
Conflicto de intereses	Los autores declaran que no existe conflicto de interés posible.
Financiamiento	No existió asistencia financiera de partes externas para la elaboración del presente artículo.
Agradecimiento	Agradezco profundamente al Padre del Cielo, a mis padres, hermanas, abuelitos y a mi hermoso sobrino Samuelito. También una mención especial para mi máxima inspiración, sin su bondad y malicia no hubiera podido entender que lo importante es la permanencia, los ciclos y la no permanencia. Al puro estilo de Erwin en la Navidad de 1925. Gracias a la ciudad del Puyo por unir nuestros caminos, el verano del 2026 lo recordaré siempre. A mis grandes colegas, Adrián y Ariel que blindaron de ideas los sentimientos que pudieron nublar mi juicio. A la comunidad gnóstica, cuyo legado prevalece a través del intelecto, regalo de la Divina Madre.
Nota	El artículo no es producto de una publicación anterior.